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Picard Iterative Method for Solving Fractional Differential Equations

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Abstract: **In this paper, based on Jumarie's modified Riemann-Liouville (R-L) fractional calculus, some examples are provided to illustrate how to use Picard iterative method to find the approximation solution of fractional differential equation. A new multiplication of fractional analytic functions plays an important role in this paper. In fact, our results are generalization of these results of ordinary differential equations.**

Keywords: **Jumarie's modified R-L fractional calculus, Picard iterative method, approximation solution, fractional differential equation, new multiplication, fractional analytic functions.**

I. INTRODUCTION

Fractional calculus is a branch of mathematical analysis, which studies several different possibilities of defining real or complex order. In the past decades, fractional calculus has developed rapidly in mathematics and applied science. Fractional calculus is very popular in many fields, such as mechanics, dynamics, control theory, physics, economics, viscoelasticity, biology, electrical engineering, etc [1-8]. However, the definition of fractional derivative is not unique. Commonly used definitions include Riemann-Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald-Letnikov (G-L) fractional derivative, Jumarie's modified R-L fractional derivative [9-14]. Since Jumarie type of R-L fractional derivative helps to avoid non-zero fractional derivative of constant function, it is easier to use this definition to connect fractional calculus with classical calculus.

In this paper, based on Jumarie type of R-L fractional calculus, we provide some examples to illustrate how to use Picard iterative method to find the approximation solution of fractional differential equation. A new multiplication of fractional analytic functions plays an important role in this article. In fact, our results are generalization of these results of ordinary differential equations.

II. PRELIMINARIES

Firstly, we introduce the fractional calculus used in this paper.

Definition 2.1 ([15])**:** Let $0 < \alpha \leq 1$, and x_0 be a real number. The Jumarie type of Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^{\alpha}}dt\ , \quad (1)$$

And the Jumarie type of Riemann-Liouville α-fractional integral is defined by

$$\left({}_{x_0}I_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}\frac{f(t)}{(x-t)^{1-\alpha}}dt\ , \quad (2)$$

where $\Gamma(\ \ )$ is the gamma function.

In the following, some properties of Jumarie type of fractional derivative are introduced.

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Proposition 2.2 ([16])**:** *If* α, β, x_0, C *are real numbers and* $\beta \geq \alpha > 0$, *then*

$$\left({}_{x_0}D_x^{\alpha}\right)\left[(x-x_0)^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-x_0)^{\beta-\alpha}, \tag{3}$$

and

$$\left({}_{x_0}D_x^{\alpha}\right)[C] = 0. \tag{4}$$

Next, the definition of fractional analytic function is introduced.

Definition 2.3 ([17])**:** Let x, x_0, and a_k be real numbers for all k, $x_0 \in (a, b)$, and $0 < \alpha \leq 1$. If the function $f_\alpha: [a, b] \to R$ can be expressed as an α-fractional power series, that is, $f_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}$ on some open interval containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0. Moreover, if $f_\alpha: [a, b] \to R$ is continuous on closed interval $[a, b]$ and it is α-fractional analytic at every point in open interval (a, b), then f_α is called an α-fractional analytic function on $[a, b]$.

In the following, we introduce a new multiplication of fractional analytic functions.

Definition 2.4 ([18])**:** If $0 < \alpha \leq 1$, and x_0 is a real number. Suppose that $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are α-fractional analytic at $x = x_0$,

$$f_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}, \tag{5}$$

$$g_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}. \tag{6}$$

Then

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes g_\alpha(x^\alpha) \\ &= \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} \otimes \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} \\ &= \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+1)}\left(\sum_{m=0}^{k} \binom{k}{m} a_{k-m} b_m\right)(x-x_0)^{k\alpha}. \end{aligned} \tag{7}$$

Equivalently,

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes g_\alpha(x^\alpha) \\ &= \sum_{k=0}^{\infty} \frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k} \otimes \sum_{k=0}^{\infty} \frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k} \\ &= \sum_{k=0}^{\infty} \frac{1}{k!}\left(\sum_{m=0}^{k} \binom{k}{m} a_{k-m} b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}. \end{aligned} \tag{8}$$

Definition 2.5 ([19])**:** Assume that $0 < \alpha \leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are α-fractional analytic at $x = x_0$,

$$f_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} = \sum_{k=0}^{\infty} \frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}, \tag{9}$$

$$g_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} = \sum_{k=0}^{\infty} \frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}. \tag{10}$$

The compositions of $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are defined by

$$(f_\alpha \circ g_\alpha)(x^\alpha) = f_\alpha\left(g_\alpha(x^\alpha)\right) = \sum_{k=0}^{\infty} \frac{a_k}{k!}\left(g_\alpha(x^\alpha)\right)^{\otimes k}, \tag{11}$$

and

$$(g_\alpha \circ f_\alpha)(x^\alpha) = g_\alpha\left(f_\alpha(x^\alpha)\right) = \sum_{k=0}^{\infty} \frac{b_k}{k!}\left(f_\alpha(x^\alpha)\right)^{\otimes k}. \tag{12}$$

Definition 2.6 ([19])**:** Suppose that $0 < \alpha \leq 1$, and x is a real number. The α-fractional exponential function is defined by

$$E_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{x^{k\alpha}}{\Gamma(k\alpha+1)} = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes k}. \tag{13}$$

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In the following, the power of fractional analytic function is defined.

Definition 2.7: Suppose that $0 < \alpha \le 1$ and n is any positive integer. Then

$$[f_\alpha(x^\alpha)]^{\otimes n} = f_\alpha(x^\alpha) \otimes \cdots \otimes f_\alpha(x^\alpha) \tag{14}$$

is called the n-th power of the α-fractional analytic function $f_\alpha(x^\alpha)$.

III. RESULTS AND EXAMPLES

In this section, the main results are provided and we give some examples to illustrate how to use Picard iterative method to find the approximation solution of fractional differential equations.

Definition 3.1: Let $0 < \alpha \le 1$, x_0 be a real number. Then the initial-value problem of α-fractional differential equation

$$\left({}_{x_0}D_x^\alpha\right)[y_\alpha(x^\alpha)] = F_\alpha\left(x^\alpha, y_\alpha(x^\alpha)\right),$$

$$y_\alpha(x_0^\alpha) = y_{0,\alpha}. \tag{15}$$

Where F_α is a continuous function in a domain A_α, and $\left(x_0^\alpha, y_{0,\alpha}\right) \in A_\alpha$. Let $\varphi_\alpha(x^\alpha)$ be a solution on an interval containing x_0. Then if $\varphi_\alpha(x^\alpha)$ is α-fractional analytic for all $x \in I$, then we have

$$\left({}_{x_0}D_x^\alpha\right)[\varphi_\alpha(x^\alpha)] = F_\alpha\left(x^\alpha, \varphi_\alpha(x^\alpha)\right),$$

$$\varphi_\alpha(x_0^\alpha) = y_{0,\alpha} \tag{16}$$

for all $x \in I$

Theorem 3.2: *Suppose that* $0 < \alpha \le 1$, x_0 *is a real number. Then* $\varphi_\alpha(x^\alpha)$ *is a solution of the initial-value problem of* α-*fractional differential equation*

$$\left({}_{x_0}D_x^\alpha\right)[y_\alpha(x^\alpha)] = F_\alpha\left(x^\alpha, y_\alpha(x^\alpha)\right),$$

$$y_\alpha(x_0^\alpha) = y_{0,\alpha}$$

if and only if $\varphi_\alpha(x^\alpha)$ *is a solution of the* α-*fractional integral equation*

$$y_\alpha(x^\alpha) = y_{0,\alpha} + \left({}_{x_0}I_x^\alpha\right)\left[F_\alpha\left(x^\alpha, y_\alpha(x^\alpha)\right)\right]. \tag{17}$$

Proof (i) Suppose that $\varphi_\alpha(x^\alpha)$ is a solution of the initial-value problem (15) on I. Then

$$\left({}_{x_0}D_x^\alpha\right)[\varphi_\alpha(x^\alpha)] = F_\alpha\left(x^\alpha, \varphi_\alpha(x^\alpha)\right) \tag{18}$$

for all $x \in I$. Since φ_α is α-fractional analytic on I and F_α is continuous in A_α, it follows that $F_\alpha\left(x^\alpha, \varphi_\alpha(x^\alpha)\right)$ is continuous on I. Thus, we obtain

$$\varphi_\alpha(x^\alpha) = y_{0,\alpha} + \left({}_{x_0}I_x^\alpha\right)\left[F_\alpha\left(x^\alpha, \varphi_\alpha(x^\alpha)\right)\right] \tag{19}$$

for all $x \in I$. Applying the initial condition $\varphi_\alpha(x_0^\alpha) = y_{0,\alpha}$, we see that φ_α is a solution of the α-fractional integral equation (17) on I.

(ii) Suppose that $\varphi_\alpha(x^\alpha)$ is a solution of the α-fractional integral equation (17) on I. Since $F_\alpha\left(x^\alpha, \varphi_\alpha(x^\alpha)\right)$ is continuous for all $x \in I$, we see that (19) holds, and by fundamental theorem of fractional integral calculus, differentiation of (19) yields

$$\left({}_{x_0}D_x^\alpha\right)[\varphi_\alpha(x^\alpha)] = F_\alpha\left(x^\alpha, \varphi_\alpha(x^\alpha)\right)$$

for all $x \in I$. We observe in (19) that $\varphi_\alpha(x_0^\alpha) = y_{0,\alpha}$. Therefore, φ_α is a solution of the initial-value problem (15).

Q.e.d.

Example 3.3: Let $0 < \alpha \le 1$. Solve the initial-value problem of α-fractional differential equation

$$\left({}_{0}D_x^\alpha\right)[y_\alpha(x^\alpha)] = \frac{1}{\Gamma(\alpha+1)} x^\alpha \otimes y_\alpha(x^\alpha),$$

$$y_\alpha(0) = 1. \tag{20}$$

Solution Since the corresponding α-fractional integral equation is

$$y_\alpha(x^\alpha) = 1 + \left({}_0I_x^\alpha\right)\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha \otimes y_\alpha(x^\alpha)\right]. \tag{21}$$

It follows that the successive approximations are given by

$$\varphi_{0,\alpha}(x^\alpha) = 1, \tag{22}$$

$$\varphi_{1,\alpha}(x^\alpha) = 1 + \left({}_0I_x^\alpha\right)\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]$$

$$= 1 + \frac{1}{2}\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes 2}, \tag{23}$$

$$\varphi_{2,\alpha}(x^\alpha) = 1 + \left({}_0I_x^\alpha\right)\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha \otimes \left\{1 + \frac{1}{2}\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes 2}\right\}\right]$$

$$= 1 + \left({}_0I_x^\alpha\right)\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha + \frac{1}{2}\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes 3}\right]$$

$$= 1 + \frac{1}{2}\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes 2} + \frac{1}{8}\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes 4}, \tag{24}$$

$$\varphi_{3,\alpha}(x^\alpha) = 1 + \left({}_0I_x^\alpha\right)\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha \otimes \left\{1 + \frac{1}{2}\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes 2} + \frac{1}{8}\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes 4}\right\}\right]$$

$$= 1 + \left({}_0I_x^\alpha\right)\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha + \frac{1}{2}\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes 3} + \frac{1}{8}\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes 5}\right]$$

$$= 1 + \frac{1}{2}\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes 2} + \frac{1}{8}\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes 4} + \frac{1}{48}\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes 6}, \tag{25}$$

and thus $\varphi_{n,\alpha}(x^\alpha)$ can be obtained by induction:

$$\varphi_{n,\alpha}(x^\alpha)$$

$$= 1 + \left(\frac{1}{2}\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes 2}\right) + \frac{1}{2!}\left(\frac{1}{2}\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes 2}\right)^{\otimes 2} + \frac{1}{3!}\left(\frac{1}{2}\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes 2}\right)^{\otimes 3} + \cdots + \frac{1}{n!}\left(\frac{1}{2}\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes}\ \right)^{\otimes n}. \tag{26}$$

It can easily be seen that the solution of this initial-value problem is

$$\varphi_\alpha(x^\alpha) = E_\alpha\left(\frac{1}{2}\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes 2}\right). \tag{27}$$

Example 3.4: If $0 < \alpha \leq 1$. Find the approximation solution of the initial-value problem of α-fractional differential equation

$$\left({}_0D_x^\alpha\right)[y_\alpha(x^\alpha)] = 2 \cdot \frac{1}{\Gamma(\alpha+1)}x^\alpha + y_\alpha(x^\alpha),$$

$$y_\alpha(0) = 1. \tag{28}$$

Solution Since

$$\varphi_{0,\alpha}(x^\alpha) = 1, \tag{29}$$

$$\varphi_{1,\alpha}(x^\alpha) = 1 + \left({}_0I_x^\alpha\right)\left[1 + 2 \cdot \frac{1}{\Gamma(\alpha+1)}x^\alpha\right]$$

$$= 1 + \frac{1}{\Gamma(\alpha+1)}x^\alpha + \left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes 2}, \tag{30}$$

$$\varphi_{2,\alpha}(x^\alpha) = 1 + \left({}_0I_x^\alpha\right)\left[2 \cdot \frac{1}{\Gamma(\alpha+1)}x^\alpha + 1 + \frac{1}{\Gamma(\alpha+1)}x^\alpha + \left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes 2}\right]$$

$$= 1 + \left({}_0I_x^\alpha\right)\left[1 + 3 \cdot \frac{1}{\Gamma(\alpha+1)}x^\alpha + \left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes 2}\right]$$

$$= 1 + \frac{1}{\Gamma(\alpha+1)}x^{\alpha} + \frac{3}{2}\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes 2} + \frac{1}{3}\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes 3}, \tag{31}$$

$$\varphi_{3,\alpha}(x^{\alpha}) = 1 + \left({}_0I_x^{\alpha}\right)\left[2\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha} + 1 + \frac{1}{\Gamma(\alpha+1)}x^{\alpha} + \frac{3}{2}\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes 2} + \frac{1}{3}\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes 3}\right]$$

$$= 1 + \left({}_0I_x^{\alpha}\right)\left[1 + 3\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha} + \frac{3}{2}\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes 2} + \frac{1}{3}\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes 3}\right]$$

$$= 1 + \frac{1}{\Gamma(\alpha+1)}x^{\alpha} + \frac{3}{2}\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes 2} + \frac{1}{2}\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes 3} + \frac{1}{12}\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes 4}, \tag{32}$$

$$\varphi_{4,\alpha}(x^{\alpha}) = 1 + \left({}_0I_x^{\alpha}\right)\left[2\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha} + 1 + \frac{1}{\Gamma(\alpha+1)}x^{\alpha} + \frac{3}{2}\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes 2} + \frac{1}{2}\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes 3} + \frac{1}{12}\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes 4}\right]$$

$$= 1 + \left({}_0I_x^{\alpha}\right)\left[1 + 3\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha} + \frac{3}{2}\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes 2} + \frac{1}{2}\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes 3} + \frac{1}{12}\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes 4}\right]$$

$$= 1 + \frac{1}{\Gamma(\alpha+1)}x^{\alpha} + \frac{3}{2}\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes 2} + \frac{1}{2}\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes 3} + \frac{1}{8}\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes 4} + \frac{1}{60}\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes 5}. \tag{33}$$

It follows that the approximation solution of this initial-value problem is

$$\varphi_{\alpha}(x^{\alpha})$$
$$= 1 + \frac{1}{\Gamma(\alpha+1)}x^{\alpha} + \frac{3}{2}\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes 2} + \frac{1}{2}\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes 3} + \frac{1}{8}\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes 4} + \frac{1}{60}\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes 5} + \cdots. \tag{34}$$

Example 3.5: Assume that $0 < \alpha \le 1$. Find the approximation solution of the initial-value problem of α-fractional differential equation

$$\left({}_0D_x^{\alpha}\right)[y_{\alpha}(x^{\alpha})] = \frac{1}{\Gamma(\alpha+1)}x^{\alpha} + [y_{\alpha}(x^{\alpha})]^{\otimes 2},$$
$$y_{\alpha}(0) = 0. \tag{35}$$

Solution

$$\varphi_{0,\alpha}(x^{\alpha}) = 0, \tag{36}$$

$$\varphi_{1,\alpha}(x^{\alpha}) = 0 + \left({}_0I_x^{\alpha}\right)\left[0 + \frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]$$

$$= \frac{1}{2}\cdot\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes 2}, \tag{37}$$

$$\varphi_{2,\alpha}(x^{\alpha}) = 0 + \left({}_0I_x^{\alpha}\right)\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha} + \left[\frac{1}{2}\cdot\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes 2}\right]^{\otimes 2}\right]$$

$$= \left({}_0I_x^{\alpha}\right)\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha} + \frac{1}{4}\cdot\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes 4}\right]$$

$$= \frac{1}{2}\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes 2} + \frac{1}{20}\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes 5}, \tag{38}$$

$$\varphi_{3,\alpha}(x^{\alpha}) = 0 + \left({}_0I_x^{\alpha}\right)\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha} + \left[\frac{1}{2}\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes 2} + \frac{1}{20}\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes 5}\right]^{\otimes 2}\right]$$

$$= \left({}_0I_x^{\alpha}\right)\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha} + \frac{1}{4}\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes 4} + \frac{1}{20}\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes 7} + \frac{1}{400}\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes 10}\right]$$

$$= \frac{1}{2}\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes 2} + \frac{1}{20}\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes 5} + \frac{1}{160}\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes 8} + \frac{1}{4400}\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes 11}, \tag{39}$$

Therefore, the approximation solution of this initial-value problem is

$$\varphi_\alpha(x^\alpha)=\frac{1}{2}\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes 2}+\frac{1}{20}\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes 5}+\frac{1}{160}\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes 8}+\frac{1}{4400}\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes 11}+\cdots. \quad (40)$$

Example 3.6: If $0<\alpha\leq 1$. Find the approximation solution of the initial-value problem of α-fractional differential equation

$$\left({}_1D_x^\alpha\right)[y_\alpha(x^\alpha)]=1+[y_\alpha(x^\alpha)]^{\otimes 3},$$
$$y_\alpha(1)=1. \quad (41)$$

Solution

$$\varphi_{0,\alpha}(x^\alpha)=1, \quad (42)$$

$$\varphi_{1,\alpha}(x^\alpha)=1+\left({}_1I_x^\alpha\right)[1+1]$$
$$=1+2\cdot\frac{1}{\Gamma(\alpha+1)}(x-1)^\alpha, \quad (43)$$

$$\varphi_{2,\alpha}(x^\alpha)$$
$$=1+\left({}_1I_x^\alpha\right)\left[1+\left[1+2\cdot\frac{1}{\Gamma(\alpha+1)}(x-1)^\alpha\right]^{\otimes 3}\right]$$
$$=1+\left({}_1I_x^\alpha\right)\left[2+6\cdot\frac{1}{\Gamma(\alpha+1)}x^\alpha+12\cdot\left[\frac{1}{\Gamma(\alpha+1)}(x-1)^\alpha\right]^{\otimes 2}+8\cdot\left[\frac{1}{\Gamma(\alpha+1)}(x-1)^\alpha\right]^{\otimes 3}\right]$$
$$=1+2\cdot\frac{1}{\Gamma(\alpha+1)}(x-1)^\alpha+3\cdot\left[\frac{1}{\Gamma(\alpha+1)}(x-1)^\alpha\right]^{\otimes 2}+4\cdot\left[\frac{1}{\Gamma(\alpha+1)}(x-1)^\alpha\right]^{\otimes 3}+2\cdot\left[\frac{1}{\Gamma(\alpha+1)}(x-1)^\alpha\right]^{\otimes 4}. \quad (44)$$

Thus, the approximation solution of this initial-value problem is

$$\varphi_\alpha(x^\alpha)$$
$$=1+2\cdot\frac{1}{\Gamma(\alpha+1)}(x-1)^\alpha+3\cdot\left[\frac{1}{\Gamma(\alpha+1)}(x-1)^\alpha\right]^{\otimes 2}+4\cdot\left[\frac{1}{\Gamma(\alpha+1)}(x-1)^\alpha\right]^{\otimes 3}+2\cdot\left[\frac{1}{\Gamma(\alpha+1)}(x-1)^\alpha\right]^{\otimes 4}+\cdots.$$
$$(45)$$

IV. CONCLUSION

In this paper, based on Jumarie type of R-L fractional calculus, we give some examples to illustrate how to use Picard iterative method to find the approximation solution of fractional differential equation. A new multiplication of fractional analytic functions plays an important role in this article. In fact, our results are generalization of these results of ordinary differential equations. In the future, we will continue to use Jumarie's modification of R-L fractional calculus to study the problems in applied mathematics and fractional differential equations.

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